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A REDUCTION OF CERTAIN DIFFERENTIAL EQUATIONS OF THE FIRST ORDER.

BY WILLIAM DUNCAN MACMILLAN.

Let us suppose there is given the differential equation

$$(1) \quad \frac{d\xi}{dt} = f(\xi),$$

where $f(\xi)$ is an analytic function of ξ which does not contain t explicitly. We propose to study the development of the function ξ in the neighborhood of the point ξ_0 , which may be an ordinary point, a pole, a zero, or a branch point of the function $f(\xi)$ but not an essential singularity. For this purpose we make the substitution $\xi - \xi_0 = x$. With this substitution the right member of (1) can be expanded in powers of x . The coefficient of the first term, x^n , can be taken equal to unity, for if it were not unity let us suppose it is $a_0 \neq 0$. After dividing the equation through by a_0 and changing the independent variable from t to τ by means of the relation $\tau = a_0 t$, the coefficient of x^n in the resulting equation will be equal to unity. The equation becomes

$$(2) \quad \frac{dx}{d\tau} = x^n + a_1 x^{n+1} + a_2 x^{n+2} + \dots$$

The constant n will be a positive or negative integer according as the point ξ_0 is a zero or a pole of $f(\xi)$; it will be zero if ξ_0 is an ordinary point. If ξ_0 is a branch point n will be a fraction, and there will be a different expansion for each branch. Choosing one of these branches a suitable transformation on x will result in a series the exponents of which are integers. For example, if $n = \frac{1}{2}$ the transformation $x^{1/2} = y$ will give a series in integral powers of y . The equation (2) can therefore be assumed to have integral exponents.

We seek now a substitution

$$(3) \quad x = y + b_2 y^2 + b_3 y^3 + \dots$$

such that the resulting differential equation in y shall be algebraic. On substituting (3) in (2) and arranging according to powers of y it is found that

$$(4) \quad \frac{dy}{d\tau} = y^n + [(n-2)b_2 + c_2]y^{n+1} + [(n-3)b_3 + c_3]y^{n+2} \\ + \dots + [(n-k)b_k + c_k]y^{n+k-1} + \dots,$$

where the b_j are the coefficients of (3) and the c_k are constants which are polynomials in the constants a_j and b_j for which the subscript $j < k$.

On equating to zero the coefficients of y^{n+1} , y^{n+2} , $\dots$, it is found that

$$(5) \quad b_k = -\frac{c_k}{n-k}, \quad k = 2, \dots, \infty.$$

Consequently if n is a negative integer, zero, or $+1$ the right member of (4) can be reduced to its first term, so that the differential equation in y is simply

$$(6) \quad y' = y^n;$$

and the transformation (3) which effects this reduction is unique. If n is zero or a negative integer, say $n = -m$, the solution of (6) is

$$(7) \quad y = [(m+1)(t-t_0)]^{1/m+1}.$$

Granting for the moment that the series (3) converges for values of y different from zero, it is seen that, if $m = 0$, the point ξ_0 is an ordinary point of $\xi(t)$; but if $m > 0$ the point ξ_0 is a branch-point of $\xi(t)$. In the first case ξ is expansible in integral powers of $(t-t_0)$ in the neighborhood of ξ_0 , and in the latter case in fractional powers.

If the exponent $n = +1$ the solution of equation (6) is

$$(8) \quad y = ce^t.$$

In this case x is expansible as a power series in e^t which approaches zero as t approaches $-\infty$. Since the point $t = \infty$ is an essential singularity of the function e^t it is clear that $t = \infty$ is also an essential singularity of the function $x(t)$.

If the exponent $n > +1$ it is seen from (5) that the particular coefficient b_n cannot be determined so as to make vanish the coefficient of y^{2n-1} in equation (4). There is no difficulty however in determining all of the other b_j , and equation (4) becomes

$$(9) \quad y' = y^n + c_n y^{2n-1}.$$

The transformation (3) is uniquely determined aside from the fact that the coefficient b_n remains arbitrary. There are therefore a single infinity of substitutions which transform (3) into (9); and it will be observed that (9) does not depend upon b_n .

The integral of equation (9) is defined by (provided $c_n \neq 0$)

$$(10) \quad -\frac{1}{c_n y^{n-1}} + \log \left(1 + \frac{1}{c_n y^{n-1}} \right) = \frac{n-1}{c_n} (t-t_0),$$

which is expressed more simply by taking

$$(11) \quad 1 + \frac{1}{c_n y^{n-1}} = W, \quad 1 - \frac{(n-1)}{c_n} (t - t_0) = \tau.$$

The result is

$$(12) \quad W - \log W = \tau,$$

which defines W as a function of τ . Let us denote this by writing $W = L(\tau)$. The function $L(\tau)$ satisfies the differential equation

$$(13) \quad \frac{dL}{d\tau} = L' = \frac{L}{L-1},$$

whence

$$(14) \quad LL' = L + L' \quad \text{and} \quad L' - 1 = \frac{1}{L-1}.$$

From (11) it is found that

$$(15) \quad c_n y^{n-1} = \frac{1}{W-1} = L' - 1;$$

so that

$$(16) \quad y = \left(\frac{L' - 1}{c_n} \right)^{1/(n-1)}$$

It appears therefore that if $n \geq 2$ and $c_n \neq 0$ the solutions of (2) are always expressible as power series in $[(L' - 1)/c_n]^{1/(n-1)}$. If c_n should be zero equation (9) becomes simply

$$y' = y^n,$$

the solution of which is

$$(17) \quad y = \frac{1}{[(n-1)(t_0 - t)]^{1/(n-1)}}.$$

It is seen from (15) that y vanishes only for $W = L(\tau) = \infty$, and from (12) it is clear that $L(\tau)$ becomes infinite only for τ equal to infinity which is an essential singularity. Hence the function $y(t)$ has an essential singularity at $t = \infty$ provided $c_n \neq 0$, and from (17) it is easy to see that if $c_n = 0$ the function $y(t)$ has merely a branch point at $t = \infty$.

Equation (12) shows that if W is negative the corresponding values of τ are complex. If W is positive but very small τ also is positive and large. As W increases to $+1$, τ decreases to $+1$; and as W increases from $+1$ to ∞ , τ also increases from $+1$ to ∞ . For purposes of computation in which y must be small the corresponding value of W must be large, and tables of the functions $L(\tau)$ and $L'(\tau)$ will be found convenient. Such tables are given and the properties of $L(\tau)$ are discussed in the following paper.

En resumé, if n is a negative integer $x = 0$ is a branch-point of the

function $x(t)$. If $n = 0$ it is an ordinary point, and if n is a positive integer it is an essential singularity, except when the constant c_n is zero in which case it is merely a branch-point.

Convergence of the series.

In order to see that the substitution (3) is a convergent one let us consider x as a function of y defined by the equation

$$(18) \quad \frac{dx}{x^n + a_1 x^{n+1} + a_2 x^{n+2} + \dots} = \frac{dy}{y^n + \beta y^{2n-1}},$$

where the constant β in the denominator of the second member is at our disposal.

If n is a negative integer or zero, $n = -m$, we can take $\beta = 0$ and expand the left member in ascending powers of x , thus:

$$[x^m + \alpha_1 x^{m+1} + \dots] dx = y^m dy.$$

Integrating we have

$$x^{m+1} + \frac{m+1}{m+2} \alpha_1 x^{m+2} + \dots = y^{m+1} + \text{const.}$$

Since y must vanish with x the constant of integration is zero. Taking now $x = y(1 + \xi)$ and dividing through by y^{m+1} , we find

$$(m+1)\xi + \frac{m+1}{m+2} \alpha_1 y + \dots = 0,$$

which has a unique and convergent solution for ξ as a power series in y for all positive values of m .

If n is unity we again take $\beta = 0$ and expand (18) with the result

$$\left[\frac{1}{x} + \alpha_0 + \alpha_1 x + \dots \right] dx = \frac{1}{y} dy.$$

Integrating we find

$$(19) \quad \log \frac{x}{y} + \alpha_0 x + \frac{1}{2} \alpha_1 x^2 + \dots = 0.$$

On taking $x = y(1 + \xi)$ it is found that (19) admits a unique solution for ξ as a power series in y vanishing with y .

If $n > +1$ we find, on expanding both members of (18), series which it is convenient to write in the form

$$\begin{aligned} \left[\frac{1}{x^n} + \frac{n-2}{n-1} \frac{\alpha_{-(n-2)}}{x^{n-1}} + \dots + \frac{\alpha_0}{(n-1)x} + \frac{\alpha_1}{(n-1)} + \frac{2\alpha_2}{(n-1)} x + \dots \right] dx \\ = \left[\frac{1}{y^n} - \frac{\beta}{y} + \beta^2 y^{n-2} - \beta^3 y^{2n-3} + \dots \right] dy. \end{aligned}$$

Integrating and multiplying through by $-(n-1)$ there results

$$\begin{aligned} \frac{1}{x^{n-1}} + \frac{\alpha_{-(n-2)}}{x^{n-2}} + \dots - \alpha_0 \log x - \alpha_1 x - \alpha_2 x^2 - \dots \\ = \gamma + \frac{1}{y^{n-1}} + (n-1)\beta \log y + \beta^2 y^{n-1} \dots, \end{aligned}$$

where γ is the constant of integration. If now we take $x = y(1 + \xi)$ it is clear that the term in $\log y$ will disappear only if $\beta = -\alpha_0/(n-1)$; and if β is given this value and we multiply through by y^{n-1} we will have

$$\begin{aligned} (20) \quad \left(\frac{1}{(1 + \xi)^{n-1}} - 1 \right) + \frac{\alpha_{-(n-2)}y}{(1 + \xi)^{n-2}} + \dots - \alpha_0 y^{n-1} \log(1 + \xi) \\ - \alpha_1 y^n (1 + \xi) - \dots] = \gamma y^{n-1} + \beta^2 y^{2n-2} + \dots \end{aligned}$$

This equation admits a unique solution for ξ as a power series in integral powers of y . The solution contains the arbitrary constant γ which appears first in the coefficient of y^{n-1} . Whatever be the value of the constant γ^* the solution will converge provided y is sufficiently small. Consequently x is expansible as a power series in y containing the arbitrary constant γ in the coefficients of y^n and higher powers. The series thus derived satisfies the same conditions as does the series (3); they are therefore identical with it and the series (3) converges.

* The series $x = y + b_2 y^2 + \dots + \gamma y^n + \dots$ when considered as a function of $(t - t_0)$ apparently has two arbitrary constants, viz., γ and t_0 . These two constants however are equivalent to only one, for we have

$$\int \frac{dx}{x^n + \dots} = \gamma + \int \frac{dy}{y^n + C_n y^{2n-1}}.$$

Differentiating both sides with respect to γ and remembering that y is not a function of γ , we find

$$\frac{\partial x / \partial \gamma}{x^n + \dots} = 1; \quad \therefore \frac{\partial x}{\partial \gamma} = x^n + \dots = \frac{\partial x}{\partial t} = -\frac{\partial x}{\partial t_0}.$$

Hence

$$\frac{\partial x}{\partial \gamma} + \frac{\partial x}{\partial t_0} \equiv 0.$$